Computation of the rotor force and flapping derivatives

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Introduction

This document provides analytic expressions of rotor forces and flapping derivatives.

The derivatives are first computed in rotor axis in non-dimensional forms and then in body-axis in dimensional forms. The later represents the rotor stability derivatives contribution for the dynamic of the gyroplane.

Nomenclature

α_F	fuselage incidence
α_S	rotor incidence relative to no-feathering plane
δ	blade profil drag
γ	lock number
$\hat{\Omega}$	non-dimensional rotor speed
$\hat{p}$	non-dimensional angular rolling velocity
$\hat{q}$	non-dimensional angular pitching velocity
$\hat{u}$	non-dimensional velocity component along $\vec{Y}_S$ axis
$\hat{w}$	non-dimensional velocity component along $\vec{Z}_S$ axis
λ	inflow ratio
λ_i	non-dimensional rotor induced velocity
μ	advance ratio
Ω	rotor speed
ρ	air density
θ_0	blade pitch at root
θ_{TW}	blade twist
a	blade lift slope
a_0	coning angle
a_1	longitudinal flapping coefficient
a_2	second order longitudinal flapping coefficient
B	tip loss factor
b	number of blade
b_1	lateral flapping coefficient
b_2	second order lateral flapping coefficient
c	blade chord
C_D^*	drag force normalized coefficient
C_H	rear force coefficient
C_L^*	lift force normalized coefficient

C_Q	rotor torque coefficient
C_T	thrust coefficient
C_{Hi}	induced rear force coefficient
C_{Hp}	profile drag rear force coefficient
C_{Qi}	induced rotor torque coefficient
C_{Qp}	profile drag rotor torque coefficient
D	rotor drag
H	total rotor force along $\vec{Y}_S$ axis equal H_i plus H_p
H_i	rotor induced force along $\vec{Y}_S$ axis positive when opposite to $\vec{Y}_S$
H_p	rotor profile drag force along $\vec{Y}_S$ axis positive when opposite to $\vec{Y}_S$
L	rotor lift
p	angular rolling velocity
Q	total rotor torque equal Q_i plus Q_p
q	angular pitching velocity
Q_i	rotor induced torque
Q_p	rotor profile drag torque
R	blade radius
T	rotor force along $\vec{Z}_S$ axis
U	aircraft velocity
u	aircraft velocity component along $\vec{Y}_S$ axis
u'	aircraft velocity component along $\vec{X}_F$ axis
v_i	rotor induced velocity
w	aircraft velocity component along $\vec{Z}_S$ axis
w'	aircraft velocity component along $\vec{Z}_F$ axis
X	rotor force along $\vec{X}_F$ axis
Z	rotor force along $\vec{Z}_F$ axis

1 Rotor axis

Figure 1 shows the rotor axis and the rotor forces. $\vec{U}$ is the aircraft velocity, $\vec{T}, \vec{H}$ the rotor thrust and rotor rear force, $\vec{L}, \vec{D}$ the lift and drag and α_S the rotor incidence relative to the no-feathering plane.

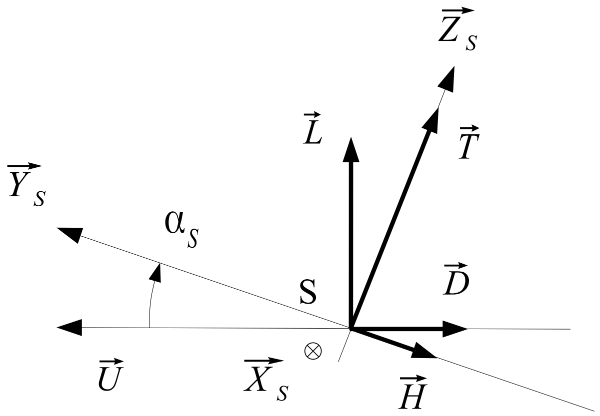

Figure 1: Rotor axis

Let us denote u and w the velocity coordinates along $\vec{Y}_S$ and $\vec{Z}_S$. We can write :

$$u = U \cos \alpha_S \tag{1}$$
$$w = -U \sin \alpha_S \tag{2}$$

2 Non-dimensionalisation of the equations

The following reference quantities are use :

- the rotor speed Ω as the unit of angular velocity ;
- the rotor tip speed ΩR as the unit of speed.

where R is the rotor radius.
Let us define the following non-dimensional quantities :

$$\hat{u} = \frac{u}{\Omega R} \tag{3}$$
$$\hat{w} = \frac{w}{\Omega R} \tag{4}$$
$$\lambda_i = \frac{v_i}{\Omega R} \tag{5}$$
$$\hat{p} = \frac{p}{\Omega} \tag{6}$$
$$\hat{q} = \frac{q}{\Omega} \tag{7}$$
$$\hat{\Omega} = \frac{\Omega}{\Omega} = 1 \tag{8}$$

where v_i is the rotor induced velocity, p the rotor axis roll rate and q the pitch rate.

3 Rotor derivatives in rotor axis

The computation of rotor derivatives in rotor axes consists to calculate derivatives of :

$$\mu, \lambda, a_0, a_1, b_1, a_2, b_2, C_T, C_{Hp}, C_{Hi}, C_{Qp}, C_{Qi}, C_L^*, C_D^* \tag{9}$$

with respects to :

$$\hat{u}, \hat{w}, \hat{p}, \hat{q}, \hat{\Omega} \tag{10}$$

where μ is the advance ratio, λ the inflow ratio, a_0, a_1, b_1, a_2, b_2 the flapping coefficients, $C_T, C_{Hp}, C_{Hi}, C_{Qp}, C_{Qi}$, respectively the thrust, profil drag rear force, induce rear force, profil drag torque and induced torque coefficients and C_L^*, C_D^* the normalized lift and drag force.

The scaling factor used to compute forces and torques coefficients are :

- $\rho bcR^3\Omega^2$ for forces ;
- $\rho bcR^4\Omega^2$ for torques ;

Note that :

$$\frac{\partial \hat{p}}{\partial \hat{\Omega}} = -\hat{p} \tag{11}$$

$$\frac{\partial \hat{q}}{\partial \hat{\Omega}} = -\hat{q} \tag{12}$$

3.1 Aircraft velocity derivatives

The velocity U is such that :

$$U^2 = u^2 + w^2 \tag{13}$$

Differentiating equation (13) with respect to u gives :

$$\frac{\partial U^2}{\partial u} = 2U\frac{\partial U}{\partial u} = 2u \tag{14}$$

Finally we get :

$$\frac{\partial U}{\partial u} = \frac{u}{U} \tag{15}$$

The same way, differentiating equation (13) with respect to w gives :

$$\frac{\partial U}{\partial w} = \frac{w}{U} \tag{16}$$

3.2 Rotor disc incidence derivative

Differentiating equation (1) with respect to α_S gives :

$$-\sin\alpha_S \frac{\partial \alpha_S}{\partial w} = -\frac{u}{U^2}\frac{\partial U}{\partial w} \tag{17}$$

As :

$$\sin\alpha_S = -\frac{w}{U} \tag{18}$$

we get :

$$\frac{\partial \alpha_S}{\partial w} = -\frac{u}{U^2} \tag{19}$$

Differentiating equation (2) with respect to α_S leads to :

$$\frac{\partial \alpha_S}{\partial u} = \frac{w}{U^2} \tag{20}$$

3.3 Advance ratio derivatives

The advance ratio μ is defined by the following expression :

$$\mu = \frac{u}{\Omega R} \tag{21}$$

We get therefore the following derivatives :

$$\frac{\partial \mu}{\partial \hat{u}} = 1 \tag{22}$$

$$\frac{\partial \mu}{\partial \hat{w}} = 0 \tag{23}$$

$$\frac{\partial \mu}{\partial \hat{q}} = 0 \tag{24}$$

$$\frac{\partial \mu}{\partial \hat{p}} = 0 \tag{25}$$

$$\frac{\partial \mu}{\partial \hat{\Omega}} = -\mu \tag{26}$$

3.4 Inflow ratio derivatives

The inflow ratio λ is defined by the following expression :

$$\lambda = \frac{w - v_i}{\Omega R} \tag{27}$$

Differentiating (27) gives the following derivatives :

$$\frac{\partial\lambda}{\partial\hat{u}} = -\frac{\partial\lambda_i}{\partial\hat{u}} \tag{28}$$

$$\frac{\partial\lambda}{\partial\hat{w}} = 1 - \frac{\partial\lambda_i}{\partial\hat{w}} \tag{29}$$

$$\frac{\partial\lambda}{\partial\hat{q}} = -\frac{\partial\lambda_i}{\partial\hat{q}} \tag{30}$$

$$\frac{\partial\lambda}{\partial\hat{p}} = -\frac{\partial\lambda_i}{\partial\hat{p}} \tag{31}$$

$$\frac{\partial\lambda}{\partial\hat{\Omega}} = -\lambda - \frac{\partial\lambda_i}{\partial\hat{\Omega}} \tag{32}$$

Partial derivatives of λ_i depends on the induced velocity model used to compute rotor induced velocity and will not be calculated in this document.

3.5 Second order longitudinal flapping angle derivatives

The expression of the a_2 coefficient is given by (see reference [1] equation (1.150)) :

$$\begin{aligned} a_2 =& \frac{\gamma\mu^2}{\gamma^2 B^8 + 144}\{-\frac{20}{3}B^3\frac{\hat{p}}{\mu} + (-\frac{128}{B\gamma} - \frac{1}{3}B^7\gamma)\frac{\hat{q}}{\mu} \\ &+ (16B + \frac{7}{108}B^9\gamma^2)\lambda + (\frac{46}{3}B^2 + \frac{7}{144}\gamma^2 B^{10})\theta_0 \\ &+ (\frac{7}{180}B^{11}\gamma^2 + 12B^3)\theta_{TW}\} \end{aligned} \tag{33}$$

Differentiating a_2 gives :

$$\begin{aligned} \frac{\partial a_2}{\partial\hat{u}} =& 2\frac{a_2}{\mu} + \frac{\gamma\mu^2}{\gamma^2 B^8 + 144}\{\frac{20}{3}B^3\frac{\hat{p}}{\mu^2} + (\frac{128}{B\gamma} + \frac{1}{3}B^7\gamma)\frac{\hat{q}}{\mu^2} \\ &+ (16B + \frac{7}{108}B^9\gamma^2)\frac{\partial\lambda}{\partial\hat{u}}\} \end{aligned} \tag{34}$$

$$\frac{\partial a_2}{\partial\hat{w}} = \frac{\gamma\mu^2}{\gamma^2 B^8 + 144}(16B + \frac{7}{108}B^9\gamma^2)\frac{\partial\lambda}{\partial\hat{w}} \tag{35}$$

$$\frac{\partial a_2}{\partial\hat{p}} = \frac{\gamma\mu^2}{\gamma^2 B^8 + 144}\{-\frac{20}{3}B^3\frac{1}{\mu} + (16B + \frac{7}{108}B^9\gamma^2)\frac{\partial\lambda}{\partial\hat{p}}\} \tag{36}$$

$$\frac{\partial a_2}{\partial\hat{q}} = \frac{\gamma\mu^2}{\gamma^2 B^8 + 144}\{-(\frac{128}{B\gamma} + \frac{1}{3}B^7\gamma)\frac{1}{\mu} + (16B + \frac{7}{108}B^9\gamma^2)\frac{\partial\lambda}{\partial\hat{q}}\} \tag{37}$$

$$\frac{\partial a_2}{\partial\hat{\Omega}} = 2\frac{a_2}{\mu}\frac{\partial\mu}{\partial\hat{\Omega}} + \frac{\gamma\mu^2}{\gamma^2 B^8 + 144}\{\frac{20}{3}B^3\frac{\hat{p}}{\mu^2}\frac{\partial\mu}{\partial\hat{\Omega}} + \frac{20}{3}B^3\frac{\hat{p}}{\mu} \tag{38}$$

$$\begin{aligned} &+ (\frac{128}{B\gamma} + \frac{1}{3}B^7\gamma)\frac{\hat{q}}{\mu^2}\frac{\partial\mu}{\partial\hat{\Omega}} + (\frac{128}{B\gamma} + \frac{1}{3}B^7\gamma)\frac{\hat{q}}{\mu} \\ &+ (16B + \frac{7}{108}B^9\gamma^2)\frac{\partial\lambda}{\partial\hat{\Omega}}\} \end{aligned} \tag{39}$$

3.6 Second order lateral flapping angle derivatives

The b_2 coefficient is given by (see reference [1] equation (1.151)) :

$$
\begin{aligned}
b_2 =& -\frac{\gamma^2\mu^2}{\gamma^2B^8+144}\{(\frac{128}{B\gamma^2}+\frac{1}{3}B^7)\frac{\hat{p}}{\mu}-\frac{20}{3}\frac{B^3}{\gamma}\frac{\hat{q}}{\mu} \\
&+\frac{5}{9}B^5\lambda+\frac{25}{36}B^6\theta_0+\frac{8}{15}B^7\theta_{TW}\}
\end{aligned} \tag{40}
$$

Differentiating b_2 gives :

$$
\begin{aligned}
\frac{\partial b_2}{\partial\hat{u}} =& 2\frac{b_2}{\mu}-\frac{\gamma^2\mu^2}{\gamma^2B^8+144}\{-(\frac{128}{\gamma^2B}+\frac{1}{3}B^7)\frac{\hat{p}}{\mu^2} \\
&+\frac{20}{3}\frac{B^3}{\gamma}\frac{\hat{q}}{\mu^2}+\frac{5}{9}B^5\frac{\partial\lambda}{\partial\hat{u}}\}
\end{aligned} \tag{41}
$$

$$
\frac{\partial b_2}{\partial\hat{w}} = -\frac{\gamma^2\mu^2}{\gamma^2B^8+144}(\frac{5}{9}B^5\frac{\partial\lambda}{\partial\hat{w}}) \tag{42}
$$

$$
\frac{\partial b_2}{\partial\hat{p}} = -\frac{\gamma^2\mu^2}{\gamma^2B^8+144}\{(\frac{128}{B\gamma^2}+\frac{1}{3}B^7)\frac{1}{\mu}+\frac{5}{9}B^5\frac{\partial\lambda}{\partial\hat{p}}\} \tag{43}
$$

$$
\frac{\partial b_2}{\partial\hat{q}} = -\frac{\gamma^2\mu^2}{\gamma^2B^8+144}\{-\frac{20}{3}\frac{B^3}{\gamma}\frac{1}{\mu}+\frac{5}{9}B^5\frac{\partial\lambda}{\partial\hat{q}}\} \tag{44}
$$

$$
\begin{aligned}
\frac{\partial b_2}{\partial\hat{\Omega}} =& 2\frac{b_2}{\mu}\frac{\partial\mu}{\partial\hat{\Omega}}-\frac{\gamma^2\mu^2}{\gamma^2B^8+144}\{ \\
&-(\frac{128}{\gamma^2B}+\frac{1}{3}B^7)\frac{\hat{p}}{\mu^2}\frac{\partial\mu}{\partial\hat{\Omega}}-(\frac{128}{\gamma^2B}+\frac{1}{3}B^7)\frac{\hat{p}}{\mu} \\
&+\frac{20}{3}\frac{B^3}{\gamma}\frac{\hat{q}}{\mu^2}\frac{\partial\mu}{\partial\hat{\Omega}}+\frac{20}{3}\frac{B^3}{\gamma}\frac{\hat{q}}{\mu}+\frac{5}{9}B^5\frac{\partial\lambda}{\partial\hat{\Omega}}\}
\end{aligned} \tag{45}
$$

3.7 Coning angle derivatives

The a_0 coefficient is given by (see reference [1] equation (1.147)) :

$$
\begin{aligned}
a_0 =& \frac{\gamma}{2}\{(\frac{1}{6}\mu B^3-\frac{5}{48}\frac{\mu^4}{\pi})\hat{p}+(\frac{1}{3}B^3+\frac{1}{4}\frac{\mu^3}{\pi})\lambda+\frac{1}{8}\mu^2B^2b_2 \\
&+(\frac{1}{4}B^4\mu^2-\frac{1}{32}\mu^4+\frac{1}{4}B^4)\theta_0+(\frac{1}{6}\mu^2B^3+\frac{1}{5}B^5)\theta_{TW}\}
\end{aligned} \tag{46}
$$

Differentiating a_0 gives :

$$
\begin{aligned}
\frac{\partial a_0}{\partial\hat{u}} =& \frac{\gamma}{2}\{(\frac{1}{6}B^3-\frac{5}{12}\frac{\mu^3}{\pi})p+\frac{3}{4}\frac{\mu^2}{\pi}\lambda+(\frac{1}{3}B^3+\frac{1}{4}\frac{\mu^3}{\pi})\frac{\partial\lambda}{\partial\hat{u}} \\
&+\frac{1}{4}B^2\mu b_2+\frac{1}{8}B^2\mu^2\frac{\partial b_2}{\partial\hat{u}} \\
&+(-\frac{1}{8}\mu^3+\frac{1}{2}B^2\mu)\theta_0+\frac{1}{3}B^3\mu\theta_{TW}\}
\end{aligned} \tag{47}
$$

$$
\frac{\partial a_0}{\partial\hat{w}} = \frac{\gamma}{2}\{(\frac{1}{3}B^3+\frac{1}{4}\frac{\mu^3}{\pi})\frac{\partial\lambda}{\partial\hat{w}}+\frac{1}{8}B^2\mu^2\frac{\partial b_2}{\partial\hat{w}} \tag{48}
$$

$$
\frac{\partial a_0}{\partial\hat{p}} = \frac{\gamma}{2}\{(\frac{1}{6}B^3\mu-\frac{5}{48}\frac{\mu^4}{\pi})+(\frac{1}{3}B^3+\frac{1}{4}\frac{\mu^3}{\pi})\frac{\partial\lambda}{\partial\hat{p}}+\frac{1}{8}B^2\mu^2\frac{\partial b_2}{\partial\hat{p}} \tag{49}
$$

$$\frac{\partial a_0}{\partial \hat{q}}=\frac{\gamma}{2}\{(\frac{1}{3}B^3+\frac{1}{4}\frac{\mu^3}{\pi})\frac{\partial \lambda}{\partial \hat{q}}+\frac{1}{8}B^2\mu^2\frac{\partial b_2}{\partial \hat{q}} \tag{50}$$

$$\begin{aligned}\frac{\partial a_0}{\partial \hat{\Omega}}=&\frac{\gamma}{2}\{(\frac{1}{6}B^3-\frac{5}{12}\frac{\mu^3}{\pi})\hat{p}\frac{\partial \mu}{\partial \hat{\Omega}}-(\frac{1}{6}B^3\mu-\frac{5}{48}\frac{\mu^4}{\pi})\hat{p}\\&+\frac{3}{4}\frac{\mu^2}{\pi}\lambda\frac{\partial \mu}{\partial \hat{\Omega}}+(\frac{1}{3}B^3+\frac{1}{4}\frac{\mu^3}{\pi})\frac{\partial \lambda}{\partial \hat{\Omega}}\\&+\frac{1}{4}B^2\mu b_2\frac{\partial \mu}{\partial \hat{\Omega}}+\frac{1}{8}B^2\mu^2\frac{\partial b_2}{\partial \hat{\Omega}}\\&+(-\frac{1}{8}\mu^3+\frac{1}{2}B^2\mu)\theta_0\frac{\partial \mu}{\partial \hat{\Omega}}\\&+\frac{1}{3}B^3\mu\theta_{TW}\frac{\partial \mu}{\partial \hat{\Omega}}\}\end{aligned} \tag{51}$$

3.8 Longitudinal flapping angle derivatives

The expression of the a_1 coefficient is given by (see reference [1] equation (1.148)) :

$$\begin{aligned}a_1=&\frac{2\mu}{B^4-\frac{1}{2}\mu^2B^2}\{(\frac{B^4}{2\mu}+\frac{5}{48}\mu^3)\hat{p}-\frac{8\hat{q}}{\gamma\mu}+(B^2-\frac{1}{4}\mu^2)\lambda\\&-\frac{1}{3}B^3b_2+(\frac{4}{3}B^3+\frac{\mu^3}{3\pi})\theta_0+B^4\theta_{TW}\}\end{aligned} \tag{52}$$

Differentiating a_1 gives :

$$\begin{aligned}\frac{\partial a_1}{\partial \hat{u}}=&\frac{2B^4+\mu^2B^2}{2\mu(B^4-\frac{1}{2}\mu^2B^2)}a_1+\frac{2\mu}{B^4-\frac{1}{2}\mu^2B^2}\{\\&(-\frac{B^4}{2\mu^2}+\frac{5}{16}\mu^2)\hat{p}+\frac{8}{\gamma}\frac{\hat{q}}{\mu^2}\\&-\frac{1}{2}\mu\lambda+(B^2-\frac{1}{4}\mu^2)\frac{\partial \lambda}{\partial \hat{u}}-\frac{B^2}{3}\frac{\partial b_2}{\partial \hat{u}}+\frac{\mu^2}{\pi}\theta_0\}\end{aligned} \tag{53}$$

$$\frac{\partial a_1}{\partial \hat{w}}=\frac{2\mu}{B^4-\frac{1}{2}\mu^2B^2}\{-\frac{B^3}{3}\frac{\partial b_2}{\partial \hat{w}}+(B^2-\frac{\mu^2}{4})\frac{\partial \lambda}{\partial \hat{w}}\} \tag{54}$$

$$\frac{\partial a_1}{\partial \hat{p}}=\frac{2\mu}{B^4-\frac{1}{2}\mu^2B^2}\{(\frac{B^4}{\mu}+\frac{5}{48}\mu^3)-\frac{B^3}{3}\frac{\partial b_2}{\partial \hat{p}}+(B^2-\frac{\mu^2}{4})\frac{\partial \lambda}{\partial \hat{p}}\} \tag{55}$$

$$\frac{\partial a_1}{\partial \hat{q}}=\frac{2\mu}{B^4-\frac{1}{2}\mu^2B^2}\{-\frac{8}{\gamma\mu}-\frac{B^3}{3}\frac{\partial b_2}{\partial \hat{q}}+(B^2-\frac{\mu^2}{4})\frac{\partial \lambda}{\partial \hat{q}}\} \tag{56}$$

$$\frac{\partial a_1}{\partial \hat{\Omega}}=\frac{2B^4+\mu^2B^2}{2\mu(B^4-\frac{1}{2}\mu^2B^2)}a_1\frac{\partial \mu}{\partial \hat{\Omega}}+\frac{2\mu}{B^4-\frac{1}{2}\mu^2B^2}\{$$

$$(-\frac{B^4}{2\mu^2}+\frac{5}{16}\mu^2)\hat{p}\frac{\partial\mu}{\partial\hat{\Omega}}-(\frac{B^4}{2\mu}+\frac{5}{48}\mu^3)\hat{p}$$
$$+\frac{8}{\gamma}\frac{\hat{q}}{\mu^2}\frac{\partial\mu}{\partial\hat{\Omega}}+\frac{8}{\gamma}\frac{\hat{q}}{\mu}-\frac{B^2}{3}\frac{\partial b_2}{\partial\hat{\Omega}}$$
$$-\frac{1}{2}\mu\lambda\frac{\partial\mu}{\partial\hat{\Omega}}+(B^2-\frac{1}{4}\mu^2)\frac{\partial\lambda}{\partial\hat{\Omega}}+\frac{\mu^2}{\pi}\theta_0\frac{\partial\mu}{\partial\hat{\Omega}}\} \quad (57)$$

3.9 Lateral flapping angle derivatives

The b_1 coefficient is given by (see reference [1] equation (1.149)) :

$$b_1=\frac{4\mu}{B^4+\frac{1}{2}\mu^2B^2}\{-\frac{4\hat{p}}{\gamma\mu}-\frac{1}{4}\frac{B^4}{\mu}\hat{q}+\frac{1}{6}B^3a_2+(\frac{1}{3}B^3+\frac{1}{9}\frac{\mu^3}{\pi})a_0\} \quad (58)$$

Differentiating b_1 gives :

$$\frac{\partial b_1}{\partial\hat{u}}=\frac{B^4+\frac{1}{2}\mu^2B^2}{\mu(B^4+\frac{1}{2}\mu^2B^2)}b_1+\frac{4\mu}{B^4+\frac{1}{2}\mu^2B^2}\{$$
$$\frac{4}{\gamma}\frac{\hat{p}}{\mu^2}+\frac{1}{4}\frac{B^4}{\mu^2}\hat{q}+\frac{1}{6}B^3\frac{\partial a_2}{\partial\hat{\mu}}+\frac{1}{3}\frac{\mu^2}{\pi}a_0$$
$$+(\frac{1}{3}B^3+\frac{1}{9}\frac{\mu^3}{\pi})\frac{\partial a_0}{\partial\hat{u}}\} \quad (59)$$

$$\frac{\partial b_1}{\partial\hat{w}}=\frac{4\mu}{B^4+\frac{1}{2}\mu^2B^2}\{\frac{1}{6}B^3\frac{\partial a_2}{\partial\hat{w}}+(\frac{1}{3}B^3+\frac{1}{9}\frac{\mu^3}{\pi})\frac{\partial a_0}{\partial\hat{w}}\} \quad (60)$$

$$\frac{\partial b_1}{\partial\hat{p}}=\frac{4\mu}{B^4+\frac{1}{2}\mu^2B^2}\{-\frac{4}{\gamma\mu}+\frac{1}{6}B^3\frac{\partial a_2}{\partial\hat{p}}+(\frac{1}{3}B^3+\frac{1}{9}\frac{\mu^3}{\pi})\frac{\partial a_0}{\partial\hat{p}}\} \quad (61)$$

$$\frac{\partial b_1}{\partial\hat{q}}=\frac{4\mu}{B^4+\frac{1}{2}\mu^2B^2}\{-\frac{1}{4}\frac{B^4}{\mu}+\frac{1}{6}B^3\frac{\partial a_2}{\partial\hat{q}}+(\frac{1}{3}B^3+\frac{1}{9}\frac{\mu^3}{\pi})\frac{\partial a_0}{\partial\hat{q}}\} \quad (62)$$

$$\frac{\partial b_1}{\partial\hat{\Omega}}=\frac{B^4+\frac{1}{2}\mu^2B^2}{\mu(B^4+\frac{1}{2}\mu^2B^2)}b_1\frac{\partial\mu}{\partial\hat{\Omega}}+\frac{4\mu}{B^4+\frac{1}{2}\mu^2B^2}\{$$
$$\frac{4}{\gamma}\frac{\hat{p}}{\mu}\frac{\partial\mu}{\partial\hat{\Omega}}+\frac{4}{\gamma\mu^2}\hat{p}+\frac{1}{4}B^4\frac{\hat{q}}{\mu}\frac{\partial\mu}{\partial\hat{\Omega}}+\frac{B^4}{4\mu}\hat{q}+\frac{1}{6}B^3\frac{\partial a_2}{\partial\hat{\Omega}}$$
$$+\frac{1}{3}\frac{\mu^2}{\pi}a_0\frac{\partial\mu}{\partial\hat{\Omega}}+(\frac{1}{3}B^3+\frac{1}{9}\frac{\mu^3}{\pi})\frac{\partial a_0}{\partial\hat{\Omega}}\} \quad (63)$$

3.10 Thrust coefficient derivatives

The value of the thrust coefficient is :

$$C_T = \frac{T}{\rho bcR^3\Omega^2} \tag{64}$$

Differentiating C_T with respect to $\hat{u}, \hat{w}, \hat{p}, \hat{q}, \hat{\Omega}$ gives :

$$\frac{\partial C_T}{\partial \hat{u}} = \frac{1}{\rho bcR^2\Omega}\frac{\partial T}{\partial u} \tag{65}$$

$$\frac{\partial C_T}{\partial \hat{w}} = \frac{1}{\rho bcR^2\Omega}\frac{\partial T}{\partial w} \tag{66}$$

$$\frac{\partial C_T}{\partial \hat{p}} = \frac{1}{\rho bcR^3\Omega}\frac{\partial T}{\partial p} \tag{67}$$

$$\frac{\partial C_T}{\partial \hat{q}} = \frac{1}{\rho bcR^3\Omega}\frac{\partial T}{\partial q} \tag{68}$$

$$\frac{\partial C_T}{\partial \hat{\Omega}} = \frac{1}{\rho bcR^3\Omega}\frac{\partial T}{\partial \hat{u}} - 2C_T \tag{69}$$

The same relations apply on C_{Hp} and C_{Hi} coefficients.

The expression of the C_T coefficient is given by (see reference [1] equation (1.158)) :

$$\begin{aligned} C_T =& \frac{a}{2}\{(-\frac{1}{16}\mu^3 + \frac{1}{4}B^2\mu)\hat{p} + (\frac{1}{4}\mu^2 + \frac{1}{2}B^2)\lambda + \frac{1}{8}\mu^3 a_1 + \frac{1}{4}B\mu^2 b_2 \\ &+ (-\frac{4}{9}\frac{\mu^3}{\pi} + \frac{1}{3}B^3 + \frac{1}{2}B\mu^2)\theta_0 + (\frac{1}{4}B^4 - \frac{1}{32}\mu^4 + \frac{1}{4}B^2\mu^2)\theta_{TW}\} \end{aligned} \tag{70}$$

Differentiating C_T gives :

$$\begin{aligned} \frac{\partial C_T}{\partial \hat{u}} =& \frac{a}{2}\{(\frac{1}{4}B^2 - \frac{3}{16}\mu^2)\hat{p} + \frac{1}{2}\mu\lambda + (\frac{1}{2}B^2 + \frac{1}{4}\mu^2)\frac{\partial \lambda}{\partial \hat{u}} \\ & \frac{3}{8}\mu^2 a_1 + \frac{1}{8}\mu^3\frac{\partial a_1}{\partial \hat{u}} + \frac{1}{2}B\mu b_2 + \frac{1}{4}B\mu^2\frac{\partial b_2}{\partial \hat{u}} \\ &+ (\frac{4}{3}\frac{\mu^2}{\pi} + B\mu)\theta_0 + (\frac{1}{2}B^2\mu - \frac{1}{8}\mu^3)\theta_{TW}\} \end{aligned} \tag{71}$$

$$\frac{\partial C_T}{\partial \hat{w}} = \frac{a}{2}\{(\frac{1}{2}B^2 + \frac{1}{4}\mu^2)\frac{\partial \lambda}{\partial \hat{w}} + \frac{1}{8}\mu^3\frac{\partial a_1}{\partial \hat{w}} + \frac{1}{4}B\mu^2\frac{\partial b_2}{\partial \hat{w}}\} \tag{72}$$

$$\begin{aligned} \frac{\partial C_T}{\partial \hat{p}} =& \frac{a}{2}\{(\frac{1}{4}B^2\mu - \frac{1}{16}\mu^3)\hat{p} + (\frac{1}{2}B^2 + \frac{1}{4}\mu^2)\frac{\partial \lambda}{\partial \hat{p}} \\ &+ \frac{1}{8}\mu^3\frac{\partial a_1}{\partial \hat{p}} + \frac{1}{4}B\mu^2\frac{\partial b_2}{\partial \hat{p}}\} \end{aligned} \tag{73}$$

$$\frac{\partial C_T}{\partial \hat{q}} = \frac{a}{2}\{(\frac{1}{2}B^2 + \frac{1}{4}\mu^2)\frac{\partial \lambda}{\partial \hat{q}} + \frac{1}{8}\mu^3\frac{\partial a_1}{\partial \hat{q}} + \frac{1}{4}B\mu^2\frac{\partial b_2}{\partial \hat{q}}\} \tag{74}$$

$$\begin{aligned} \frac{\partial C_T}{\partial \hat{\Omega}} =& \frac{a}{2}\{(\frac{1}{4}B^2 - \frac{3}{16}\mu^2)\hat{p}\frac{\partial \mu}{\partial \hat{\Omega}} - (\frac{1}{4}B^2\mu - \frac{3}{16}\mu^3)\hat{p} \\ &+ \frac{1}{2}\mu\lambda\frac{\partial \mu}{\partial \hat{\Omega}} + (\frac{1}{2}B^2 + \frac{1}{4}\mu^2)\frac{\partial \lambda}{\partial \hat{\Omega}} \end{aligned}$$

$$+\frac{3}{8}\mu^2 a_1\frac{\partial\mu}{\partial\hat{\Omega}}+\frac{1}{8}\mu^3\frac{\partial a_1}{\partial\hat{\Omega}}+\frac{1}{2}B\mu b_2\frac{\partial\mu}{\partial\hat{\Omega}}+\frac{1}{4}B\mu^2\frac{\partial b_2}{\partial\hat{\Omega}}$$
$$+(\frac{4}{3}\frac{\mu^2}{\pi}+B\mu)\theta_0\frac{\partial\mu}{\partial\hat{\Omega}}+(\frac{1}{2}B^2\mu-\frac{1}{8}\mu^3)\theta_{TW}\frac{\partial\mu}{\partial\hat{\Omega}}\} \quad (75)$$

3.11 Rear force coefficient derivatives

The expression of the C_{Hp} and C_{Hi} coefficient are given by (see reference [1] equation (1.159) and (1.160)) :

$$C_{Hp}=\frac{1}{4}\delta\mu \quad (76)$$
$$C_{Hi}=\frac{a}{2}\{(-\frac{1}{2}B^2\lambda-\frac{1}{6}B^3\theta_0-\frac{1}{8}B^4\theta_{TW}+\frac{5}{12}B^3b_2-\frac{1}{16}B^2\mu a_1)\hat{p}$$
$$+(-\frac{1}{6}B^3a_0+\frac{5}{12}B^3a_2-\frac{1}{16}B^2\mu b_1)\hat{q}$$
$$+(\frac{3}{4}B^2a_1-\frac{1}{2}B\theta_0\mu-\frac{1}{4}B\mu b_2-\frac{1}{4}B^2\mu\theta_{TW})\lambda$$
$$+(\frac{1}{3}B^3a_1+\frac{3}{8}B^2\mu b_2)\theta_0+(\frac{1}{4}B^4a_1+\frac{1}{4}B^3\mu b_2)\theta_{TW}$$
$$+(-\frac{1}{6}B^3b_1-\frac{1}{2}B^2\mu a_2)a_0$$
$$+\frac{1}{4}B^2\mu a0^2+\frac{1}{4}B^2\mu a_1^2-\frac{1}{4}B^3b_2a_1+\frac{1}{4}B^3b_1a_2\} \quad (77)$$

Differentiating C_{Hp} with respect to $\hat{u}$ gives :

$$\frac{\partial C_{Hp}}{\partial\hat{u}}=\frac{1}{4}\delta \quad (78)$$

Differentiating C_{Hp} with respect to $\hat{\Omega}$ gives :

$$\frac{\partial C_{Hp}}{\partial\hat{\Omega}}=\frac{1}{4}\delta\frac{\partial\mu}{\partial\hat{\Omega}} \quad (79)$$

Differentiating C_{Hi} gives :

$$\frac{\partial C_{Hi}}{\partial\hat{u}}=\frac{a}{2}\{(-\frac{1}{2}B^2\frac{\partial\lambda}{\partial\hat{u}}+\frac{5}{12}B^3\frac{\partial b_2}{\partial\hat{u}}-\frac{1}{16}B^2a_1-\frac{1}{16}B^2\mu\frac{\partial a_1}{\partial\hat{u}})\hat{p}$$
$$+(-\frac{1}{6}B^3\frac{\partial a_0}{\partial\hat{u}}+\frac{5}{12}B^2\frac{\partial a_2}{\partial\hat{u}}-\frac{1}{16}B^2b_1-\frac{1}{16}B^2\mu\frac{\partial b_1}{\partial\hat{u}})\hat{q}$$
$$+(\frac{3}{4}B^2\frac{\partial a_1}{\partial\hat{u}}-\frac{1}{2}B\theta_0-\frac{1}{4}B\mu-\frac{1}{4}B\mu\frac{\partial b_2}{\partial\hat{u}}-\frac{1}{4}B^2\theta_{TW})\lambda$$
$$+(\frac{3}{4}B^2a_1-\frac{1}{2}B\theta_0\mu-\frac{1}{4}B\mu b_2-\frac{1}{4}B^2\mu\theta_{TW})\frac{\partial\lambda}{\partial\hat{u}}$$
$$+(\frac{1}{3}B^3\frac{\partial a_1}{\partial\hat{u}}+\frac{3}{8}B^2b_2+\frac{3}{8}B^2\mu\frac{\partial b_2}{\partial\hat{u}})\theta_0$$

$$+ (\frac{1}{4}B^4\frac{\partial a_1}{\partial \hat{u}} + \frac{1}{4}B^3 b_2 + \frac{1}{4}B^3\mu\frac{\partial b_2}{\partial \hat{u}})\theta_{TW}$$

$$+ (-\frac{1}{6}B^3\frac{\partial b_1}{\partial \hat{u}} - \frac{1}{2}B^3 a_2 - \frac{1}{2}B^3\mu\frac{\partial a_2}{\partial \hat{u}})a_0$$

$$+ (-\frac{1}{6}B^3 b_1 - \frac{1}{2}B^3\mu a_2)\frac{\partial a_0}{\partial \hat{u}}$$

$$+ \frac{1}{4}B^2 a_0^2 + \frac{1}{2}B^2\mu a0\frac{\partial a_0}{\partial \hat{u}} + \frac{1}{4}B^2 a_1^2 + \frac{1}{2}B^2\mu a_1\frac{\partial a_1}{\partial \hat{u}}$$

$$- \frac{1}{4}B^3 b_2\frac{\partial a_1}{\partial \hat{u}} - \frac{1}{4}B^3 a_1\frac{\partial b_2}{\partial \hat{u}} + \frac{1}{4}B^3 b_1\frac{\partial a_2}{\partial \hat{u}} + \frac{1}{4}B^3 a_2\frac{\partial 1_1}{\partial \hat{u}}\} \quad (80)$$

$$\frac{\partial C_{Hi}}{\partial \hat{w}} = \frac{a}{2}\{(-\frac{1}{2}B^2\frac{\partial \lambda}{\partial \hat{w}} + \frac{5}{12}B^3\frac{\partial b_2}{\partial \hat{w}} - \frac{1}{16}B^2\mu\frac{\partial a_1}{\partial \hat{w}})\hat{p}$$

$$+ (-\frac{1}{6}B^3\frac{\partial a_0}{\partial \hat{w}} + \frac{5}{12}B^2\frac{\partial a_2}{\partial \hat{w}} - \frac{1}{16}B^2\mu\frac{\partial b_1}{\partial \hat{w}})\hat{q}$$

$$+ (\frac{3}{4}B^2\frac{\partial a_1}{\partial \hat{w}} - \frac{1}{4}B\mu\frac{\partial b_2}{\partial \hat{w}})\lambda$$

$$+ (\frac{3}{4}B^2 a_1 - \frac{1}{2}B\theta_0\mu - \frac{1}{4}B\mu b_2 - \frac{1}{4}B^2\mu\theta_{TW})\frac{\partial \lambda}{\partial \hat{w}}$$

$$+ (\frac{1}{3}B^3\frac{\partial a_1}{\partial \hat{w}} + \frac{3}{8}B^2\mu\frac{\partial b_2}{\partial \hat{w}})\theta_0 + (\frac{1}{4}B^4\frac{\partial a_1}{\partial \hat{w}} + \frac{1}{4}B^3\mu\frac{\partial b_2}{\partial \hat{w}})\theta_{TW}$$

$$+ (-\frac{1}{6}B^3\frac{\partial b_1}{\partial \hat{w}} - \frac{1}{2}B^3\mu\frac{\partial a_2}{\partial \hat{w}})a_0 + (-\frac{1}{6}B^3 b_1 - \frac{1}{2}B^3\mu a_2)\frac{\partial a_0}{\partial \hat{w}}$$

$$+ \frac{1}{2}B^2\mu a_0\frac{\partial a_0}{\partial \hat{w}} + \frac{1}{2}B^2\mu a_1\frac{\partial a_1}{\partial \hat{w}}$$

$$- \frac{1}{4}B^3 b_2\frac{\partial a_1}{\partial \hat{w}} - \frac{1}{4}B^3 a_1\frac{\partial b_2}{\partial \hat{w}} + \frac{1}{4}B^3 b_1\frac{\partial a_2}{\partial \hat{w}} + \frac{1}{4}B^3 a_2\frac{\partial b_1}{\partial \hat{w}}\} \quad (81)$$

$$\frac{\partial C_{Hi}}{\partial \hat{p}} = \frac{a}{2}\{(-\frac{1}{2}B^2\lambda - \frac{1}{6}B^3\theta_0 - \frac{1}{8}B^4\theta_{TW} + \frac{5}{12}B^3 b_2 - \frac{1}{16}B^2\mu a_1)$$

$$+ (-\frac{1}{2}B^2\frac{\partial \lambda}{\partial \hat{p}} + \frac{5}{12}B^3\frac{\partial b_2}{\partial \hat{p}} - \frac{1}{16}B^2\mu\frac{\partial a_1}{\partial \hat{p}})\hat{p}$$

$$+ (-\frac{1}{6}B^3\frac{\partial a_0}{\partial \hat{p}} + \frac{5}{12}B^2\frac{\partial a_2}{\partial \hat{p}} - \frac{1}{16}B^2\mu\frac{\partial b_1}{\partial \hat{p}})\hat{q}$$

$$+ (\frac{3}{4}B^2\frac{\partial a_1}{\partial \hat{p}} - \frac{1}{4}B\mu\frac{\partial b_2}{\partial \hat{p}})\lambda$$

$$+ (\frac{3}{4}B^2 a_1 - \frac{1}{2}B\theta_0\mu - \frac{1}{4}B\mu b_2 - \frac{1}{4}B^2\mu\theta_{TW})\frac{\partial \lambda}{\partial \hat{p}}$$

$$+ (\frac{1}{3}B^3\frac{\partial a_1}{\partial \hat{p}} + \frac{3}{8}B^2\mu\frac{\partial b_2}{\partial \hat{p}})\theta_0 + (\frac{1}{4}B^4\frac{\partial a_1}{\partial \hat{p}} + \frac{1}{4}B^3\mu\frac{\partial b_2}{\partial \hat{p}})\theta_{TW}$$

$$+ (-\frac{1}{6}B^3\frac{\partial b_1}{\partial \hat{p}} - \frac{1}{2}B^3\mu\frac{\partial a_2}{\partial \hat{p}})a_0 + (-\frac{1}{6}B^3 b_1 - \frac{1}{2}B^3\mu a_2)\frac{\partial a_0}{\partial \hat{p}}$$

$$+ \frac{1}{2}B^2\mu a_0\frac{\partial a_0}{\partial \hat{p}} + \frac{1}{2}B^2\mu a_1\frac{\partial a_1}{\partial \hat{p}}$$

$$
-\frac{1}{4}B^3 b_2 \frac{\partial a_1}{\partial \hat{p}} - \frac{1}{4}B^3 a_1 \frac{\partial b_2}{\partial \hat{p}} + \frac{1}{4}B^3 b_1 \frac{\partial a_2}{\partial \hat{p}} + \frac{1}{4}B^3 a_2 \frac{\partial b_1}{\partial \hat{p}}\} \quad (82)
$$

$$
\begin{aligned}
\frac{\partial C_{Hi}}{\partial \hat{q}} = & \frac{a}{2}\{(-\frac{1}{6}B^3 a_0 + \frac{5}{12}B^2 a_2 - \frac{1}{16}B^2 \mu b_1) \\
& + (-\frac{1}{2}B^2 \frac{\partial \lambda}{\partial \hat{q}} + \frac{5}{12}B^3 \frac{\partial b_2}{\partial \hat{q}} - \frac{1}{16}B^2 \mu \frac{\partial a_1}{\partial \hat{q}})\hat{q} \\
& + (-\frac{1}{6}B^3 \frac{\partial a_0}{\partial \hat{q}} + \frac{5}{12}B^2 \frac{\partial a_2}{\partial \hat{q}} - \frac{1}{16}B^2 \mu \frac{\partial b_1}{\partial \hat{q}})\hat{q} \\
& + (\frac{3}{4}B^2 \frac{\partial a_1}{\partial \hat{q}} - \frac{1}{4}B\mu \frac{\partial b_2}{\partial \hat{q}})\lambda \\
& + (\frac{3}{4}B^2 a_1 - \frac{1}{2}B\theta_0 \mu - \frac{1}{4}B\mu b_2 - \frac{1}{4}B^2 \mu \theta_{TW})\frac{\partial \lambda}{\partial \hat{q}} \\
& + (\frac{1}{3}B^3 \frac{\partial a_1}{\partial \hat{q}} + \frac{3}{8}B^2 \mu \frac{\partial b_2}{\partial \hat{q}})\theta_0 + (\frac{1}{4}B^4 \frac{\partial a_1}{\partial \hat{q}} + \frac{1}{4}B^3 \mu \frac{\partial b_2}{\partial \hat{q}})\theta_{TW} \\
& + (-\frac{1}{6}B^3 \frac{\partial b_1}{\partial \hat{q}} - \frac{1}{2}B^3 \mu \frac{\partial a_2}{\partial \hat{q}})a_0 + (-\frac{1}{6}B^3 b_1 - \frac{1}{2}B^3 \mu a_2)\frac{\partial a_0}{\partial \hat{q}} \\
& + \frac{1}{2}B^2 \mu a_0 \frac{\partial a_0}{\partial \hat{q}} + \frac{1}{2}B^2 \mu a_1 \frac{\partial a_1}{\partial \hat{q}} \\
& - \frac{1}{4}B^3 b_2 \frac{\partial a_1}{\partial \hat{q}} - \frac{1}{4}B^3 a_1 \frac{\partial b_2}{\partial \hat{q}} + \frac{1}{4}B^3 b_1 \frac{\partial a_2}{\partial \hat{q}} + \frac{1}{4}B^3 a_2 \frac{\partial b_1}{\partial \hat{q}}\} \quad (83)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C_{Hi}}{\partial \hat{\Omega}} = & \frac{a}{2}\{(-\frac{1}{2}B^2 \frac{\partial \lambda}{\partial \hat{\Omega}} + \frac{5}{12}B^3 \frac{\partial b_2}{\partial \hat{\Omega}} - \frac{1}{16}B^2 a_1 \frac{\partial \mu}{\partial \hat{\Omega}} - \frac{1}{16}B^2 \mu \frac{\partial a_1}{\partial \hat{\Omega}})\hat{p} \\
& - (-\frac{1}{2}B^2 \lambda - \frac{1}{6}B^3 \theta_0 - \frac{1}{8}B^4 \theta_{TW} + \frac{5}{12}B^3 b_2 - \frac{1}{16}B^2 \mu a_1)\hat{p} \\
& + (-\frac{1}{6}B^3 \frac{\partial a_0}{\partial \hat{\Omega}} + \frac{5}{12}B^2 \frac{\partial a_2}{\partial \hat{\Omega}} - \frac{1}{16}B^2 b_1 \frac{\partial \mu}{\partial \hat{\Omega}} - \frac{1}{16}B^2 \mu \frac{\partial b_1}{\partial \hat{\Omega}})\hat{q} \\
& - (-\frac{1}{6}B^3 a_0 + \frac{5}{12}B^2 a_2 - \frac{1}{16}B^2 \mu b_1)\hat{q} \\
& + (\frac{3}{4}B^2 \frac{\partial a_1}{\partial \hat{\Omega}} - \frac{1}{2}B\theta_0 \frac{\partial \mu}{\partial \hat{\Omega}} - \frac{1}{4}B\mu \frac{\partial \mu}{\partial \hat{\Omega}} - \frac{1}{4}B\mu \frac{\partial b_2}{\partial \hat{\Omega}} - \frac{1}{4}B^2 \theta_{TW} \frac{\partial \mu}{\partial \hat{\Omega}})\lambda \\
& + (\frac{3}{4}B^2 a_1 - \frac{1}{2}B\theta_0 \mu - \frac{1}{4}B\mu b_2 - \frac{1}{4}B^2 \mu \theta_{TW})\frac{\partial \lambda}{\partial \hat{\Omega}} \\
& + (\frac{1}{3}B^3 \frac{\partial a_1}{\partial \hat{\Omega}} + \frac{3}{8}B^2 b_2 \frac{\partial \mu}{\partial \hat{\Omega}} + \frac{3}{8}B^2 \mu \frac{\partial b_2}{\partial \hat{\Omega}})\theta_0 \\
& + (\frac{1}{4}B^4 \frac{\partial a_1}{\partial \hat{\Omega}} + \frac{1}{4}B^3 b_2 \frac{\partial \mu}{\partial \hat{\Omega}} + \frac{1}{4}B^3 \mu \frac{\partial b_2}{\partial \hat{\Omega}})\theta_{TW} \\
& + (-\frac{1}{6}B^3 \frac{\partial b_1}{\partial \hat{\Omega}} - \frac{1}{2}B^3 a_2 \frac{\partial \mu}{\partial \hat{\Omega}} - \frac{1}{2}B^3 \mu \frac{\partial a_2}{\partial \hat{\Omega}})a_0 \\
& + (-\frac{1}{6}B^3 b_1 - \frac{1}{2}B^3 \mu a_2)\frac{\partial a_0}{\partial \hat{\Omega}}
\end{aligned}
$$

$$+\frac{1}{4}B^2a_0^2\frac{\partial \mu}{\partial \hat{\Omega}}+\frac{1}{2}B^2\mu a0\frac{\partial a_0}{\partial \hat{\Omega}}+\frac{1}{4}B^2a_1^2\frac{\partial \mu}{\partial \hat{\Omega}}+\frac{1}{2}B^2\mu a_1\frac{\partial a_1}{\partial \hat{\Omega}}$$
$$-\frac{1}{4}B^3b_2\frac{\partial a_1}{\partial \hat{\Omega}}-\frac{1}{4}B^3a_1\frac{\partial b_2}{\partial \hat{\Omega}}+\frac{1}{4}B^3b_1\frac{\partial a_2}{\partial \hat{\Omega}}+\frac{1}{4}B^3a_2\frac{\partial 1_1}{\partial \hat{\Omega}}\} \quad (84)$$

3.12 Rotor torque coefficient derivatives

Expressions of rotor torque coefficients are :

$$C_{Qp}=\frac{Q_p}{\rho bcR^4\Omega^2} \quad (85)$$
$$C_{Qi}=\frac{Q_i}{\rho bcR^4\Omega^2} \quad (86)$$

Differentiating C_{Qp} and C_{Qi} with respect to $\hat{u},\hat{w},\hat{p},\hat{q},\hat{\Omega}$ gives :

$$\frac{\partial C_Q}{\partial \hat{u}}=\frac{1}{\rho bcR^3\Omega}\frac{\partial Q}{\partial u} \quad (87)$$
$$\frac{\partial C_Q}{\partial \hat{w}}=\frac{1}{\rho bcR^3\Omega}\frac{\partial Q}{\partial w} \quad (88)$$
$$\frac{\partial C_Q}{\partial \hat{p}}=\frac{1}{\rho bcR^4\Omega}\frac{\partial Q}{\partial p} \quad (89)$$
$$\frac{\partial C_Q}{\partial \hat{q}}=\frac{1}{\rho bcR^4\Omega}\frac{\partial Q}{\partial q} \quad (90)$$
$$\frac{\partial C_Q}{\partial \hat{\Omega}}=\frac{1}{\rho bcR^4\Omega}\frac{\partial Q}{\partial \Omega}-2C_Q \quad (91)$$

where C_Q, Q are either C_{Qp}, Q_p or C_{Qi}, Q_i.

The values of the profile drag torque coefficients C_{Qp} and C_{Qi} are given by the following expressions (see reference [1] equation (1.162) and (1.163)) :

$$C_{Qp}=\frac{1}{64}\delta\{-8-8\mu^2+\mu^4\} \quad (92)$$
$$C_{Qi}=\frac{a}{2}\{K_{p^2}\hat{p}^2+K_p\hat{p}+K_{q^2}\hat{q}^2+K_q\hat{q}+K_{\lambda_2}\lambda^2+K_\lambda\lambda+K_{a_2}a_2+K_{b_2}b_2$$
$$+K_{a_0^2}a_0^2+K_{a_1^2}a_1^2+K_{b_1^2}b_1^2+K\} \quad (93)$$

with :

$$K_{p^2}=-\frac{5}{64}\mu^4+\frac{1}{8}B^4 \quad (94)$$
$$K_p=-\frac{4}{45}\frac{\mu^4\theta_0}{\pi}+\frac{1}{8}B^4\mu\theta_{TW}+\frac{1}{6}B^3\mu\theta_0+\frac{1}{6}B^3\mu b_2-\frac{1}{4}B^4a_1+\frac{1}{4}\mu^3\lambda \quad (95)$$
$$K_{q^2}=-\frac{1}{64}\mu^4+\frac{1}{8}B^4 \quad (96)$$
$$K_q=\frac{8}{45}\frac{\mu^4a_0}{\pi}+\frac{1}{6}B^3\mu a_2-\frac{1}{3}B^3\mu a_0+\frac{1}{4}B^4b_1 \quad (97)$$

$$K_{\lambda_2} = \frac{1}{2}B^2 - \frac{1}{4}\mu^2 \tag{98}$$

$$K_\lambda = \frac{1}{32}\mu^4\theta_{TW} + \frac{1}{2}B^2\mu a_1 + \frac{1}{4}B^4\theta_{TW} + \frac{1}{3}B^3\theta_0 + \frac{2}{9}\frac{\mu^3\theta_0}{\pi} - \frac{3}{8}\mu^3 a_1 \tag{99}$$

$$K_{a_2} = -\frac{1}{4}B^2\mu^2 a_0 - \frac{1}{6}B^3\mu b_1 + \frac{1}{2}B^4 a_2 \tag{100}$$

$$K_{b_2} = \frac{1}{8}B^2\theta_0\mu^2 + \frac{1}{12}B^3\mu^2\theta_{TW} + \frac{1}{6}B^3\mu a_1 + \frac{1}{2}B^4 b_2 \tag{101}$$

$$K_{a_0^2} = \frac{1}{4}B^2\mu^2 - \frac{1}{16}\mu^4 \tag{102}$$

$$K_{a_1^2} = \frac{1}{8}B^4 + \frac{3}{16}B^2\mu^2 \tag{103}$$

$$K_{b_1^2} = \frac{1}{8}B^4 + \frac{1}{16}B^2\mu^2 \tag{104}$$

$$K = -\frac{1}{3}B^3\mu a_0 b_1 \tag{105}$$

Derivative of C_{Qp} with respect to $\hat{u}$ gives :

$$\frac{\partial C_{Qp}}{\partial \hat{u}} = \frac{1}{16}\delta\{-4\mu + \mu^3\} \tag{106}$$

Derivative of C_{Qp} with respect to $\hat{\Omega}$ gives :

$$\frac{\partial C_{Qp}}{\partial \hat{\Omega}} = \frac{1}{16}\delta\{-4\mu + \mu^3\}\frac{\partial \mu}{\partial \hat{\Omega}} \tag{107}$$

Derivative of C_{Qi} with respect to $\hat{u}$ gives :

$$\begin{aligned}\frac{\partial C_{Qp}}{\partial \hat{u}} =& \frac{\partial K_{p^2}}{\partial \hat{u}}\hat{p}^2 + \frac{\partial K_p}{\partial \hat{u}}\hat{p} + \frac{\partial K_{q^2}}{\partial \hat{u}}\hat{q}^2 + \frac{\partial K_q}{\partial \hat{u}}\hat{q} \\ &+ \frac{\partial K_{\lambda_2}}{\partial \hat{u}}\lambda_2 + 2K_{\lambda_2}\lambda\frac{\partial \lambda}{\partial \hat{u}} + \frac{\partial K_\lambda}{\partial \hat{u}}\lambda + K_\lambda\frac{\partial \lambda}{\partial \hat{u}} \\ &+ \frac{\partial K_{a_2}}{\partial \hat{u}}a_2 + K_{a_2}\frac{\partial a_2}{\partial \hat{u}} + \frac{\partial K_{b_2}}{\partial \hat{u}}b_2 + K_{b_2}\frac{\partial b_2}{\partial \hat{u}} \\ &+ \frac{\partial K_{a_0^2}}{\partial \hat{u}}a_0^2 + 2K_{a_0^2}a_0\frac{\partial a_0}{\partial \hat{u}} + \frac{\partial K_{a_1^2}}{\partial \hat{u}}a_1^2 + 2K_{a_1^2}a_1\frac{\partial a_1}{\partial \hat{u}} \\ &+ \frac{\partial K_{b_1^2}}{\partial \hat{u}}a_0^2 + 2K_{b_1^2}b_1\frac{\partial a_0}{\partial \hat{u}} + \frac{\partial K}{\partial \hat{u}}\end{aligned} \tag{108}$$

with :

$$\frac{\partial K_{p^2}}{\partial \hat{u}} = -\frac{5}{16}\mu^3 \tag{109}$$

$$\begin{aligned}\frac{\partial K_p}{\partial \hat{u}} =& -\frac{16}{45}\frac{\theta_0}{\pi}\mu^3 + \frac{1}{8}B^4\theta_{TW} + \frac{1}{6}B^3\theta_0 + \frac{1}{6}B^3 b_2 + \frac{1}{6}B^3\mu\frac{\partial b_2}{\partial \hat{u}} \\ &- \frac{1}{4}B^4\frac{\partial a_1}{\partial \hat{u}} + \frac{3}{4}\mu^2\lambda + \frac{1}{4}\mu^3\frac{\partial \lambda}{\partial \hat{u}}\end{aligned} \tag{110}$$

$$\frac{\partial K_{q^2}}{\partial \hat{u}} = -\frac{1}{16}\mu^3 \tag{111}$$

$$\begin{aligned}\frac{\partial K_q}{\partial \hat{u}} =& \frac{32}{45}\frac{a_0}{\pi}\mu^3 + \frac{1}{6}B^3 a_2 + \frac{1}{6}B^3\mu\frac{\partial a_2}{\partial \hat{u}} - \frac{1}{3}B^3 a_0 - \frac{1}{3}B^3\mu\frac{\partial a_0}{\partial \hat{u}}\\ &+ \frac{1}{4}B^4\frac{\partial b_1}{\partial \hat{u}}\end{aligned} \tag{112}$$

$$\frac{\partial K_{\lambda_2}}{\partial \hat{u}} = -\frac{1}{2}\mu \tag{113}$$

$$\begin{aligned}\frac{\partial K_\lambda}{\partial \hat{u}} =& \frac{1}{8}\mu^3\theta_{TW} + \frac{1}{2}B^2 a_1 + \frac{1}{2}B^2\mu\frac{\partial a_1}{\partial \hat{u}} + \frac{2}{3}\frac{\theta_0}{\pi}\mu^2\\ &- \frac{9}{8}\mu^2 a_1 - \frac{3}{8}\mu^3\frac{\partial a_1}{\partial \hat{u}}\end{aligned} \tag{114}$$

$$\frac{\partial K_{a_2}}{\partial \hat{u}} = -\frac{1}{2}B^2\mu a_0 - \frac{1}{4}B^2\mu^2\frac{\partial a_0}{\partial \hat{u}} - \frac{1}{6}B^3 b_1 - \frac{1}{6}B^3\mu\frac{\partial b_1}{\partial \hat{u}} + \frac{1}{2}B^4\frac{\partial a_2}{\partial \hat{u}} \tag{115}$$

$$\frac{\partial K_{b_2}}{\partial \hat{u}} = \frac{1}{4}B^2\theta_0\mu + \frac{1}{6}B^3\mu\theta_{TW} + \frac{1}{6}B^3 a_1 + \frac{1}{6}B^3\mu\frac{\partial a_1}{\partial \hat{u}} + \frac{1}{2}B^4\frac{\partial b_2}{\partial \hat{u}} \tag{116}$$

$$\frac{\partial K_{a_0^2}}{\partial \hat{u}} = \frac{1}{2}B^2\mu - \frac{1}{4}\mu^3 \tag{117}$$

$$\frac{\partial K_{a_1^2}}{\partial \hat{u}} = \frac{3}{8}B^2\mu \tag{118}$$

$$\frac{\partial K_{b_1^2}}{\partial \hat{u}} = \frac{1}{8}B^2\mu \tag{119}$$

$$\frac{\partial K}{\partial \hat{u}} = -\frac{1}{3}B^3 a_0 b_1 - \frac{1}{3}B^3\mu b_1\frac{\partial a_0}{\partial \hat{u}} - \frac{1}{3}B^3\mu a_0\frac{\partial b_1}{\partial \hat{u}} \tag{120}$$

Derivative of C_{Qi} with respect to $\hat{w}$ gives :

$$\begin{aligned}\frac{\partial C_{Qp}}{\partial \hat{w}} =& \frac{\partial K_p}{\partial \hat{w}}\hat{p} + \frac{\partial K_q}{\partial \hat{w}}\hat{q}\\ &+ 2K_{\lambda_2}\lambda\frac{\partial \lambda}{\partial \hat{w}} + \frac{\partial K_\lambda}{\partial \hat{w}}\lambda + K_\lambda\frac{\partial \lambda}{\partial \hat{w}}\\ &+ K_{a_2}\frac{\partial a_2}{\partial \hat{w}} + K_{b_2}\frac{\partial b_2}{\partial \hat{w}}\\ &+ 2K_{a_0^2}a_0\frac{\partial a_0}{\partial \hat{w}} + 2K_{a_1^2}a_1\frac{\partial a_1}{\partial \hat{w}} + 2K_{b_1^2}b_1\frac{\partial a_0}{\partial \hat{w}} + \frac{\partial K}{\partial \hat{w}}\end{aligned} \tag{121}$$

with :

$$\frac{\partial K_p}{\partial \hat{w}} = \frac{1}{6}B^3\mu\frac{\partial b_2}{\partial \hat{w}} - \frac{1}{4}B^4\frac{\partial a_1}{\partial \hat{w}} + \frac{1}{4}\mu^3\frac{\partial \lambda}{\partial \hat{w}} \tag{122}$$

$$\frac{\partial K_q}{\partial \hat{w}} = \frac{1}{6}B^3\mu\frac{\partial a_2}{\partial \hat{w}} - \frac{1}{3}B^3\mu\frac{\partial a_0}{\partial \hat{w}} + \frac{1}{4}B^4\frac{\partial b_1}{\partial \hat{w}} \tag{123}$$

$$\frac{\partial K_\lambda}{\partial \hat{w}} = \frac{1}{2}B^2\mu\frac{\partial a_1}{\partial \hat{w}} - \frac{3}{8}\mu^3\frac{\partial a_1}{\partial \hat{w}} \tag{124}$$

$$\frac{\partial K_{a_2}}{\partial \hat{w}} = -\frac{1}{4}B^2\mu^2\frac{\partial a_0}{\partial \hat{w}} - \frac{1}{6}B^3\mu\frac{\partial b_1}{\partial \hat{w}} + \frac{1}{2}B^4\frac{\partial a_2}{\partial \hat{w}} \tag{125}$$

$$\frac{\partial K_{b_2}}{\partial \hat{w}} = \frac{1}{6}B^3\mu\frac{\partial a_1}{\partial \hat{w}} + \frac{1}{2}B^4\frac{\partial b_2}{\partial \hat{w}} \tag{126}$$

$$\frac{\partial K}{\partial \hat{w}} = -\frac{1}{3}B^3\mu b_1\frac{\partial a_0}{\partial \hat{w}} - \frac{1}{3}B^3\mu a_0\frac{\partial b_1}{\partial \hat{w}} \tag{127}$$

Derivative of C_{Qi} with respect to $\hat{q}$ gives :

$$\begin{aligned}\frac{\partial C_{Qp}}{\partial \hat{q}} =& \frac{\partial K_p}{\partial \hat{q}}\hat{p} + \frac{\partial K_q}{\partial \hat{q}}\hat{q} + 2K_{q^2}q + K_q \\ &+ 2K_{\lambda_2}\lambda\frac{\partial \lambda}{\partial \hat{q}} + \frac{\partial K_\lambda}{\partial \hat{q}}\lambda + K_\lambda\frac{\partial \lambda}{\partial \hat{q}} \\ &+ K_{a_2}\frac{\partial a_2}{\partial \hat{q}} + K_{b_2}\frac{\partial b_2}{\partial \hat{q}} \\ &+ 2K_{a_0^2}a_0\frac{\partial a_0}{\partial \hat{q}} + 2K_{a_1^2}a_1\frac{\partial a_1}{\partial \hat{q}} + 2K_{b_1^2}b_1\frac{\partial a_0}{\partial \hat{q}} + \frac{\partial K}{\partial \hat{q}}\end{aligned} \tag{128}$$

with :

$$\frac{\partial K_p}{\partial \hat{q}} = \frac{1}{6}B^3\mu\frac{\partial b_2}{\partial \hat{q}} - \frac{1}{4}B^4\frac{\partial a_1}{\partial \hat{q}} + \frac{1}{4}\mu^3\frac{\partial \lambda}{\partial \hat{q}} \tag{129}$$

$$\frac{\partial K_q}{\partial \hat{q}} = \frac{1}{6}B^3\mu\frac{\partial a_2}{\partial \hat{q}} - \frac{1}{3}B^3\mu\frac{\partial a_0}{\partial \hat{q}} + \frac{1}{4}B^4\frac{\partial b_1}{\partial \hat{q}} \tag{130}$$

$$\frac{\partial K_\lambda}{\partial \hat{q}} = \frac{1}{2}B^2\mu\frac{\partial a_1}{\partial \hat{q}} - \frac{3}{8}\mu^3\frac{\partial a_1}{\partial \hat{q}} \tag{131}$$

$$\frac{\partial K_{a_2}}{\partial \hat{q}} = -\frac{1}{4}B^2\mu^2\frac{\partial a_0}{\partial \hat{q}} - \frac{1}{6}B^3\mu\frac{\partial b_1}{\partial \hat{q}} + \frac{1}{2}B^4\frac{\partial a_2}{\partial \hat{q}} \tag{132}$$

$$\frac{\partial K_{b_2}}{\partial \hat{q}} = \frac{1}{6}B^3\mu\frac{\partial a_1}{\partial \hat{q}} + \frac{1}{2}B^4\frac{\partial b_2}{\partial \hat{q}} \tag{133}$$

$$\frac{\partial K}{\partial \hat{q}} = -\frac{1}{3}B^3\mu b_1\frac{\partial a_0}{\partial \hat{q}} - \frac{1}{3}B^3\mu a_0\frac{\partial b_1}{\partial \hat{q}} \tag{134}$$

Derivative of C_{Qi} with respect to $\hat{p}$ gives :

$$\begin{aligned}\frac{\partial C_{Qp}}{\partial \hat{p}} =& \frac{\partial K_p}{\partial \hat{p}}\hat{p} + 2K_{p^2}p + K_p + \frac{\partial K_q}{\partial \hat{p}}\hat{p} \\ &+ 2K_{\lambda_2}\lambda\frac{\partial \lambda}{\partial \hat{p}} + \frac{\partial K_\lambda}{\partial \hat{p}}\lambda + K_\lambda\frac{\partial \lambda}{\partial \hat{p}} \\ &+ K_{a_2}\frac{\partial a_2}{\partial \hat{p}} + K_{b_2}\frac{\partial b_2}{\partial \hat{p}} \\ &+ 2K_{a_0^2}a_0\frac{\partial a_0}{\partial \hat{p}} + 2K_{a_1^2}a_1\frac{\partial a_1}{\partial \hat{p}} + 2K_{b_1^2}b_1\frac{\partial a_0}{\partial \hat{p}} + \frac{\partial K}{\partial \hat{p}}\end{aligned} \tag{135}$$

with :

$$\frac{\partial K_p}{\partial \hat{p}} = \frac{1}{6}B^3\mu\frac{\partial b_2}{\partial \hat{p}} - \frac{1}{4}B^4\frac{\partial a_1}{\partial \hat{p}} + \frac{1}{4}\mu^3\frac{\partial \lambda}{\partial \hat{p}} \tag{136}$$

$$\frac{\partial K_q}{\partial \hat{p}} = \frac{1}{6}B^3\mu\frac{\partial a_2}{\partial \hat{p}} - \frac{1}{3}B^3\mu\frac{\partial a_0}{\partial \hat{p}} + \frac{1}{4}B^4\frac{\partial b_1}{\partial \hat{p}} \tag{137}$$

$$\frac{\partial K_\lambda}{\partial \hat{p}} = \frac{1}{2}B^2\mu\frac{\partial a_1}{\partial \hat{p}} - \frac{3}{8}\mu^3\frac{\partial a_1}{\partial \hat{p}} \tag{138}$$

$$\frac{\partial K_{a_2}}{\partial \hat{p}} = -\frac{1}{4}B^2\mu^2\frac{\partial a_0}{\partial \hat{p}} - \frac{1}{6}B^3\mu\frac{\partial b_1}{\partial \hat{p}} + \frac{1}{2}B^4\frac{\partial a_2}{\partial \hat{p}} \tag{139}$$

$$\frac{\partial K_{b_2}}{\partial \hat{p}} = \frac{1}{6}B^3\mu\frac{\partial a_1}{\partial \hat{p}} + \frac{1}{2}B^4\frac{\partial b_2}{\partial \hat{p}} \tag{140}$$

$$\frac{\partial K}{\partial \hat{p}} = -\frac{1}{3}B^3\mu b_1\frac{\partial a_0}{\partial \hat{p}} - \frac{1}{3}B^3\mu a_0\frac{\partial b_1}{\partial \hat{p}} \tag{141}$$

Derivative of C_{Qi} with respect to $\hat{\Omega}$ gives :

$$\begin{aligned}\frac{\partial C_{Qp}}{\partial \hat{\Omega}} =& \frac{\partial K_{p^2}}{\partial \hat{\Omega}}\hat{p}^2 + \frac{\partial K_p}{\partial \hat{\Omega}}\hat{p} + \frac{\partial K_{q^2}}{\partial \hat{\Omega}}\hat{q}^2 + \frac{\partial K_q}{\partial \hat{\Omega}}\hat{q} \\ &+ \frac{\partial K_{\lambda_2}}{\partial \hat{\Omega}}\lambda_2 + 2K_{\lambda_2}\lambda\frac{\partial \lambda}{\partial \hat{\Omega}} + \frac{\partial K_\lambda}{\partial \hat{\Omega}}\lambda + K_\lambda\frac{\partial \lambda}{\partial \hat{\Omega}} \\ &+ \frac{\partial K_{a_2}}{\partial \hat{\Omega}}a_2 + K_{a_2}\frac{\partial a_2}{\partial \hat{\Omega}} + \frac{\partial K_{b_2}}{\partial \hat{\Omega}}b_2 + K_{b_2}\frac{\partial b_2}{\partial \hat{\Omega}} \\ &+ \frac{\partial K_{a_0^2}}{\partial \hat{\Omega}}a_0^2 + 2K_{a_0^2}a_0\frac{\partial a_0}{\partial \hat{\Omega}} + \frac{\partial K_{a_1^2}}{\partial \hat{\Omega}}a_1^2 + 2K_{a_1^2}a_1\frac{\partial a_1}{\partial \hat{\Omega}} \\ &+ \frac{\partial K_{b_1^2}}{\partial \hat{\Omega}}a_0^2 + 2K_{b_1^2}b_1\frac{\partial a_0}{\partial \hat{\Omega}} + \frac{\partial K}{\partial \hat{\Omega}}\end{aligned} \tag{142}$$

with :

$$\frac{\partial K_{p^2}}{\partial \hat{\Omega}} = -\frac{5}{16}\mu^3\frac{\partial \mu}{\partial \hat{\Omega}} \tag{143}$$

$$\begin{aligned}\frac{\partial K_p}{\partial \hat{\Omega}} =& \left(-\frac{16}{45}\frac{\theta_0}{\pi}\mu^3 + \frac{1}{8}B^4\theta_{TW} + \frac{1}{6}B^3\theta_0 + \frac{1}{6}B^3b_2 + \frac{3}{4}\mu^2\lambda\right)\frac{\partial \mu}{\partial \hat{\Omega}} \\ &+ \frac{1}{6}B^3\mu\frac{\partial b_2}{\partial \hat{\Omega}} - \frac{1}{4}B^4\frac{\partial a_1}{\partial \hat{\Omega}} + \frac{1}{4}\mu^3\frac{\partial \lambda}{\partial \hat{\Omega}}\end{aligned} \tag{144}$$

$$\frac{\partial K_{q^2}}{\partial \hat{\Omega}} = -\frac{1}{16}\mu^3\frac{\partial \mu}{\partial \hat{\Omega}} \tag{145}$$

$$\begin{aligned}\frac{\partial K_q}{\partial \hat{\Omega}} =& \left(\frac{32}{45}\frac{a_0}{\pi}\mu^3 + \frac{1}{6}B^3a_2 - \frac{1}{3}B^3a_0\right)\frac{\partial \mu}{\partial \hat{\Omega}} \\ &+ \frac{1}{6}B^3\mu\frac{\partial a_2}{\partial \hat{\Omega}} - \frac{1}{3}B^3\mu\frac{\partial a_0}{\partial \hat{\Omega}} + \frac{1}{4}B^4\frac{\partial b_1}{\partial \hat{\Omega}}\end{aligned} \tag{146}$$

$$\frac{\partial K_{\lambda_2}}{\partial \hat{\Omega}} = -\frac{1}{2}\mu\frac{\partial \mu}{\partial \hat{\Omega}} \tag{147}$$

$$\begin{aligned}\frac{\partial K_\lambda}{\partial \hat{\Omega}} =& \left(\frac{1}{8}\mu^3\theta_{TW} + \frac{1}{2}B^2a_1 + \frac{2}{3}\frac{\theta_0}{\pi}\mu^2 - \frac{9}{8}\mu^2a_1\right)\frac{\partial \mu}{\partial \hat{\Omega}} \\ &+ \frac{1}{2}B^2\mu\frac{\partial a_1}{\partial \hat{\Omega}} - \frac{3}{8}\mu^3\frac{\partial a_1}{\partial \hat{\Omega}}\end{aligned} \tag{148}$$

$$\frac{\partial K_{a_2}}{\partial \hat{\Omega}} = \left(-\frac{1}{2}B^2\mu a_0 - \frac{1}{6}B^3b_1\right)\frac{\partial \mu}{\partial \hat{\Omega}} - \frac{1}{4}B^2\mu^2\frac{\partial a_0}{\partial \hat{\Omega}} - \frac{1}{6}B^3\mu\frac{\partial b_1}{\partial \hat{\Omega}} + \frac{1}{2}B^4\frac{\partial a_2}{\partial \hat{\Omega}} \tag{149}$$

$$\frac{\partial K_{b_2}}{\partial \hat{\Omega}} = \left(\frac{1}{4}B^2\theta_0\mu + \frac{1}{6}B^3\mu\theta_{TW} + \frac{1}{6}B^3a_1\right)\frac{\partial \mu}{\partial \hat{\Omega}} + \frac{1}{6}B^3\mu\frac{\partial a_1}{\partial \hat{\Omega}} + \frac{1}{2}B^4\frac{\partial b_2}{\partial \hat{\Omega}} \tag{150}$$

$$\frac{\partial K_{a_0^2}}{\partial\hat{\Omega}} = (\frac{1}{2}B^2\mu - \frac{1}{4}\mu^3)\frac{\partial\mu}{\partial\hat{\Omega}} \tag{151}$$

$$\frac{\partial K_{a_1^2}}{\partial\hat{\Omega}} = \frac{3}{8}B^2\mu\frac{\partial\mu}{\partial\hat{\Omega}} \tag{152}$$

$$\frac{\partial K_{b_1^2}}{\partial\hat{\Omega}} = \frac{1}{8}B^2\mu\frac{\partial\mu}{\partial\hat{\Omega}} \tag{153}$$

$$\frac{\partial K}{\partial\hat{\Omega}} = -\frac{1}{3}B^3 a_0 b_1\frac{\partial\mu}{\partial\hat{\Omega}} - \frac{1}{3}B^3\mu b_1\frac{\partial a_0}{\partial\hat{\Omega}} - \frac{1}{3}B^3\mu a_0\frac{\partial b_1}{\partial\hat{\Omega}} \tag{154}$$

3.13 Rotor lift and drag derivatives

The lift and drag forces are given by :

$$L = T\cos\alpha_S - H\sin\alpha_S \tag{155}$$

$$D = T\sin\alpha_S + H\cos\alpha_S \tag{156}$$

The normalized lift and drag force are defined by :

$$C_L^* = \frac{L}{\rho bcR^3\Omega^2} \tag{157}$$

$$C_D^* = \frac{D}{\rho bcR^3\Omega^2} \tag{158}$$

Differentiating theses coefficients with respect to $\hat{u}, \hat{w}, \hat{p}, \hat{q}, \hat{\Omega}$ gives :

$$\frac{\partial C_L^*}{\partial\hat{u}} = \frac{1}{\rho bcR^2\Omega}\frac{\partial L}{\partial u} \tag{159}$$

$$\frac{\partial C_L^*}{\partial\hat{w}} = \frac{1}{\rho bcR^2\Omega}\frac{\partial L}{\partial w} \tag{160}$$

$$\frac{\partial C_L^*}{\partial\hat{p}} = \frac{1}{\rho bcR^3\Omega}\frac{\partial L}{\partial p} \tag{161}$$

$$\frac{\partial C_L^*}{\partial\hat{q}} = \frac{1}{\rho bcR^3\Omega}\frac{\partial L}{\partial q} \tag{162}$$

$$\frac{\partial C_L^*}{\partial\hat{\Omega}} = \frac{1}{\rho bcR^3\Omega}\frac{\partial L}{\partial\Omega} - 2C_L^* \tag{163}$$

and :

$$\frac{\partial C_D^*}{\partial\hat{u}} = \frac{1}{\rho bcR^2\Omega}\frac{\partial D}{\partial u} \tag{164}$$

$$\frac{\partial C_D^*}{\partial\hat{w}} = \frac{1}{\rho bcR^2\Omega}\frac{\partial D}{\partial w} \tag{165}$$

$$\frac{\partial C_D^*}{\partial\hat{p}} = \frac{1}{\rho bcR^3\Omega}\frac{\partial D}{\partial p} \tag{166}$$

$$\frac{\partial C_D^*}{\partial\hat{q}} = \frac{1}{\rho bcR^3\Omega}\frac{\partial D}{\partial q} \tag{167}$$

$$\frac{\partial C_D^*}{\partial\hat{\Omega}} = \frac{1}{\rho bcR^3\Omega}\frac{\partial D}{\partial\Omega} - 2C_D^* \tag{168}$$

The relations between normalized lift, normalized drag force and thrust, rear force coefficients are :

$$C_L^* = C_T \cos\alpha_S - C_H \sin\alpha_S \tag{169}$$
$$C_D^* = C_T \sin\alpha_S + C_H \cos\alpha_S \tag{170}$$

Theses relations lead to the following derivatives :

$$\frac{\partial C_L^*}{\partial \hat{u}} = \frac{\partial C_T}{\partial \hat{u}} \cos\alpha_S - C_T \sin\alpha_S \frac{\partial \alpha_S}{\partial \hat{u}} - \frac{\partial C_H}{\partial \hat{u}} \sin\alpha_S - C_H \cos\alpha_S \frac{\partial \alpha_S}{\partial \hat{u}} \tag{171}$$
$$\frac{\partial C_L^*}{\partial \hat{p}} = \frac{\partial C_T}{\partial \hat{p}} \cos\alpha_S - \frac{\partial C_H}{\partial \hat{p}} \sin\alpha_S \tag{172}$$
$$\frac{\partial C_L^*}{\partial \hat{q}} = \frac{\partial C_T}{\partial \hat{q}} \cos\alpha_S - \frac{\partial C_H}{\partial \hat{q}} \sin\alpha_S \tag{173}$$
$$\frac{\partial C_L^*}{\partial \hat{\Omega}} = \frac{\partial C_T}{\partial \hat{\Omega}} \cos\alpha_S - \frac{\partial C_H}{\partial \hat{\Omega}} \sin\alpha_S - 2C_L^* \tag{174}$$

and

$$\frac{\partial C_D^*}{\partial \hat{u}} = \frac{\partial C_T}{\partial \hat{u}} \sin\alpha_S + C_T \cos\alpha_S \frac{\partial \alpha_S}{\partial \hat{u}} + \frac{\partial C_H}{\partial \hat{u}} \cos\alpha_S - C_H \sin\alpha_S \frac{\partial \alpha_S}{\partial \hat{u}} \tag{175}$$
$$\frac{\partial C_D^*}{\partial \hat{p}} = \frac{\partial C_T}{\partial \hat{p}} \sin\alpha_S + \frac{\partial C_H}{\partial \hat{p}} \cos\alpha_S \tag{176}$$
$$\frac{\partial C_D^*}{\partial \hat{q}} = \frac{\partial C_T}{\partial \hat{q}} \sin\alpha_S + \frac{\partial C_H}{\partial \hat{q}} \cos\alpha_S \tag{177}$$
$$\frac{\partial C_D^*}{\partial \hat{\Omega}} = \frac{\partial C_T}{\partial \hat{\Omega}} \sin\alpha_S + \frac{\partial C_H}{\partial \hat{\Omega}} \cos\alpha_S - 2C_D^* \tag{178}$$

3.14 Derivatives with respect to U and α_S

The derivatives of forces and flapping have been computed with respect to $\hat{u}$ and $\hat{w}$. It is more convenient some times to compute the derivatives with respect to U and α_S.

We can write :

$$\frac{\partial}{\partial U} = \frac{\partial}{\partial \hat{u}} \frac{\partial \hat{u}}{\partial U} + \frac{\partial}{\partial \hat{w}} \frac{\partial \hat{w}}{\partial U} \tag{179}$$

Definition of $\hat{u}$ and $\hat{w}$ leads to :

$$\frac{\partial}{\partial U} = \frac{1}{U}[\hat{u}\frac{\partial}{\partial \hat{u}} + \hat{w}\frac{\partial}{\partial \hat{w}}] \tag{180}$$

The same way we get :

$$\frac{\partial}{\partial \alpha_S} = \hat{w}\frac{\partial}{\partial \hat{u}} - \hat{u}\frac{\partial}{\partial \hat{w}} \tag{181}$$

3.15 Derivatives with respect to μ and $\hat{w}$

Let us compute now derivatives with respect to μ and $\hat{w}$ where the first is calculated for constant α_S and the second for constant U :

$$\left.\frac{\partial}{\partial \hat{w}}\right)_{const.U} = \frac{\partial}{\partial \hat{u}} \left.\frac{\partial \hat{u}}{\partial \hat{w}}\right)_{const.U} + \left.\frac{\partial}{\partial \hat{w}}\right)_{const.\hat{u}} \left.\frac{\partial \hat{w}}{\partial \hat{w}}\right)_{const.U} \tag{182}$$

Let us denote $x = \sin \alpha_S$. From :

$$\hat{u} = \frac{U}{\Omega R}\sqrt{1 - \sin^2 \alpha_S} \tag{183}$$

we get :

$$\frac{\partial \hat{u}}{\partial x} = -\frac{U}{\Omega R} \tan \alpha_S \tag{184}$$

Therefore :

$$\left.\frac{\partial}{\partial \hat{w}}\right)_{const.U} = -\tan \alpha_S \frac{\partial}{\partial \hat{u}} + \left.\frac{\partial}{\partial \hat{w}}\right)_{const.\hat{u}} \tag{185}$$

Furthermore :

$$\left.\frac{\partial}{\partial \mu}\right)_{const.\alpha_S} = \frac{\partial}{\partial \hat{u}} \left.\frac{\partial \hat{u}}{\partial \mu}\right)_{const.\alpha_S} + \left.\frac{\partial}{\partial \hat{w}}\right)_{const.\hat{u}} \left.\frac{\partial \hat{w}}{\partial \mu}\right)_{const.\alpha_S} \tag{186}$$

Let us denote this time $x = \cos \alpha_S$. From :

$$\hat{w} = -\frac{U}{\Omega R}\sqrt{1 - \cos^2 \alpha_S} \tag{187}$$

we get :

$$\frac{\partial \hat{w}}{\partial x} = \frac{U}{\Omega R} \tan \alpha_S \tag{188}$$

Therefore :

$$\left.\frac{\partial}{\partial \mu}\right)_{const.\alpha_S} = \frac{\partial}{\partial \hat{u}} + \tan \alpha_S \left.\frac{\partial}{\partial \hat{w}}\right)_{const.\hat{u}} \tag{189}$$

4 Force and rotor torque derivatives in fuselage axis

Fuselages axis $\vec{X}_F$ and $\vec{Z}_F$ are defined in figure 2.

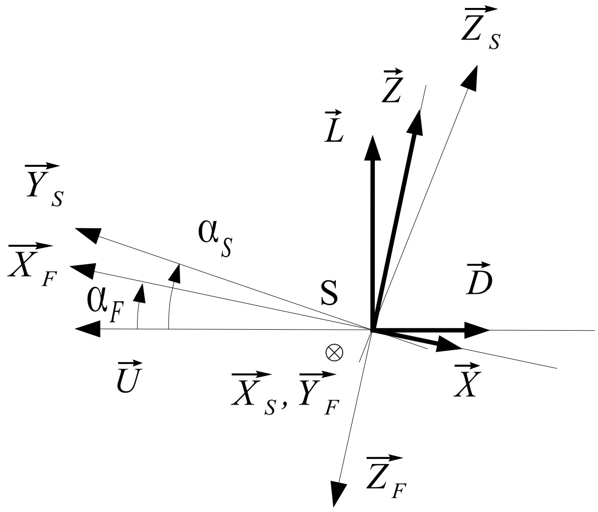

Figure 2: Fuselage axis

α_F is the fuselage incidence.

Lest us define u' and w' the velocity coordinates in body axis. We get :

$$u' = U \cos \alpha_F \tag{190}$$
$$w' = U \sin \alpha_F \tag{191}$$

The computation of derivatives in body axis consists to calculate the derivatives of :

$$X, Z, Q \tag{192}$$

with respects to :

$$u', w', q, \Omega \tag{193}$$

where X and Z are respectively the rotor forces along $\vec{X}_F$ and $\vec{Z}_F$ axis.

From figure 2, we can write :

$$X = \frac{1}{U}(-u'D + w'L) \tag{194}$$
$$Z = \frac{1}{U}(-w'D - u'L) \tag{195}$$

Let us denote $\Delta\alpha$:

$$\Delta\alpha = \alpha_S - \alpha_F \tag{196}$$

From :

$$u = U \cos(\alpha_F + \Delta\alpha) \tag{197}$$
$$w = -U \sin(\alpha_F + \Delta\alpha) \tag{198}$$

we can write at first order in $\Delta\alpha$:

$$u = u' \cos \Delta\alpha - w' \sin \Delta\alpha \tag{199}$$
$$w = -w' \cos \Delta\alpha - u' \sin \Delta\alpha \tag{200}$$

and then we get :

$$\frac{\partial u}{\partial u'} = \cos\Delta\alpha \tag{201}$$
$$\frac{\partial u}{\partial w'} = -\sin\Delta\alpha \tag{202}$$
$$\frac{\partial w}{\partial u'} = \sin\Delta\alpha \tag{203}$$
$$\frac{\partial w}{\partial w'} = -\cos\Delta\alpha \tag{204}$$

Differentiating L, D with respect to u' and w' gives :

$$\frac{\partial L}{\partial u'} = \frac{\partial L}{\partial u}\cos\Delta\alpha - \frac{\partial L}{\partial w}\sin\Delta\alpha \tag{205}$$
$$\frac{\partial L}{\partial w'} = -\frac{\partial L}{\partial u}\sin\Delta\alpha - \frac{\partial L}{\partial w}\cos\Delta\alpha \tag{206}$$
$$\frac{\partial D}{\partial u'} = \frac{\partial D}{\partial u}\cos\Delta\alpha - \frac{\partial D}{\partial w}\sin\Delta\alpha \tag{207}$$
$$\frac{\partial D}{\partial w'} = -\frac{\partial D}{\partial u}\sin\Delta\alpha - \frac{\partial D}{\partial w}\cos\Delta\alpha \tag{208}$$

4.1 X force derivative

Derivatives of X with respect to u', w', q, Ω are computed from equation (194) and leads to :

$$\frac{\partial X}{\partial u'} = \frac{u'}{U^3}(u'D - w'L) + \frac{1}{U}(-D - u'\frac{\partial D}{\partial u'} + w'\frac{\partial L}{\partial u'}) \tag{209}$$
$$\frac{\partial X}{\partial w'} = \frac{w'}{U^3}(u'D - w'L) + \frac{1}{U}(-u'\frac{\partial D}{\partial w'} + L + w'\frac{\partial L}{\partial w'}) \tag{210}$$
$$\frac{\partial X}{\partial q} = \frac{1}{U}(-u'\frac{\partial D}{\partial q} + w'\frac{\partial L}{\partial q}) \tag{211}$$
$$\frac{\partial X}{\partial \Omega} = \frac{1}{U}(-u'\frac{\partial D}{\partial \Omega} + w'\frac{\partial L}{\partial \Omega}) \tag{212}$$

4.2 Z force derivative

Derivatives of Z with respect to u', w', q, Ω is computed from equation (195) and leads to :

$$\frac{\partial Z}{\partial u'} = \frac{u'}{U^3}(w'D + u'L) - \frac{1}{U}(w'\frac{\partial D}{\partial u'} + L + u'\frac{\partial L}{\partial u'}) \tag{213}$$
$$\frac{\partial Z}{\partial w'} = \frac{w'}{U^3}(w'D + u'L) - \frac{1}{U}(D + w'\frac{\partial D}{\partial w'} + u'\frac{\partial L}{\partial w'}) \tag{214}$$
$$\frac{\partial Z}{\partial q} = \frac{1}{U}(-w'\frac{\partial D}{\partial q} - u'\frac{\partial L}{\partial q}) \tag{215}$$
$$\frac{\partial Z}{\partial \Omega} = \frac{1}{U}(-w'\frac{\partial D}{\partial \Omega} - u'\frac{\partial L}{\partial \Omega}) \tag{216}$$

4.3 Q torque derivative

Q derivatives are finally expressed as :

$$\frac{\partial Q}{\partial u'} = \frac{\partial Q}{\partial u} \cos \Delta\alpha - \frac{\partial Q}{\partial w} \sin \Delta\alpha \tag{217}$$

$$\frac{\partial Q}{\partial w'} = -\frac{\partial Q}{\partial u} \sin \Delta\alpha - \frac{\partial Q}{\partial w} \cos \Delta\alpha \tag{218}$$

References

[1] J. Fourcade : Calcul des caractéristiques aérodynamiques d'un rotor en mouvement de translation rectiligne et rotation uniformes ; `www.volucres.fr` ; <*download*>